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Leptogenesis and bi-unitary parametrization of neutrino Yukawa matrix

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Abstract. We analyze the neutrino Yukawa matrix by considering three constraints: the out-of-equilibrium condition of the lepton number-violating process responsible for leptogenesis, the upper bound of the branching ratio of the lepton flavor violating decay, and the prediction of large mixing angles using the see-saw mechanism. In a certain parametrization with a bi-unitary transformation, it is shown that the structure which satisfies the constraints can be characterized by only seven types of Yukawa matrices. The constraint of the branching ratio of LFV turns out to be redundant after applying the other two constraints. We propose that this parametrization can be the framework in which the CP asymmetry of a lepton number-violating process can be predicted in terms of observable neutrino parameters at low energy, if necessary, under assumptions following from a theory with additional symmetries. There is an appealing model of the neutrino Yukawa matrix considering the CP asymmetry for leptogenesis, giving a theoretical motivation to reduce the number of free parameters.

1 Introduction: Leptogenesis

The observed baryon asymmetry in the universe (BAU) can be explained with a process that satisfies three conditions: baryon number (B) violation, charge conjugation (C) violation and charge conjugation and parity (CP) violation, and the B-violating process should be out of equilibrium [1]. Leptogenesis is a scenario for BAU in which the initial lepton asymmetry is recycled into a baryon asymmetry by the sphaleron process [2]. The initial lepton asymmetry can originate from loop-level processes involving the Yukawa couplings of Majorana neutrinos [3]. The baryon asymmetry ΔB can be generated after the sphaleron process in thermal equilibrium washes $\Delta(B+L)$ out of $\Delta(B-L)+\Delta(B+L)$, where the $\Delta(B-L)$ is equivalent to the initial lepton asymmetry $-\Delta L$ [2,4,5]. The constraints on the chemical potentials of particles in thermal equilibrium establish the relations among different asymmetries. The amount of baryon asymmetry is described in terms of the ratio of particle number density to entropy density, i.e., baryon number density with respect to a comoving volume element, by

$$Y_B = \frac{(n_B - n_{\overline{B}})}{s}, \tag{1}$$

with entropy s, the range of which is [6]

$$Y_B \approx (0.6\text{–}1)\times 10^{-10}. \tag{2}$$

The baryon asymmetry is related to the lepton asymmetry [5,7] by

$$Y_B = aY_{B-L} = \frac{a}{a-1}Y_L, \tag{3}$$

where

$$a \equiv \frac{8N_F + 4N_H}{22N_F + 13N_H}, \tag{4}$$

for example, $a = 28/79$ for the standard model (SM) with three generations of fermions and a single Higgs doublet, $N_F = 3, N_H = 1$, and $a = 8/23$ if $N_F = 3, N_H = 2$ as in supersymmetric models. Thus, the amount of original lepton asymmetry should not be less than $\sim 10^{-9}$ in order to provide the necessary amount of baryon asymmetry.

The generation of a lepton asymmetry also requires the CP-asymmetry and out-of-equilibrium condition. Y_L is explicitly parameterized by two factors, ϵ_i, the size of CP asymmetry, and κ, the dilution factor from the wash-out process. We have

$$Y_L = \frac{(n_L - n_{\overline{L}})}{s} = \kappa\frac{\epsilon_i}{g^*}, \tag{5}$$

where $g^* \simeq 110$ is the number of relativistic degrees of freedom. The ϵ_i is the magnitude of CP asymmetry in decays of heavy Majorana neutrinos [8,9],

$$\epsilon_i = \frac{\Gamma(\nu_{\rm R}\to \ell H) - \Gamma(\nu_{\rm R}\to \ell^c H^c)}{\Gamma(\nu_{\rm R}\to \ell H) + \Gamma(\nu_{\rm R}\to \ell^c H^c)}, \tag{6}$$

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where i is 1 to 3, labeling the generation. When one generation of the right neutrinos has a mass far below the masses for the other generations, i.e., $M_1 \ll M_2, M_3$, the ϵ_i in (6) reduces to ϵ_1 from the decay of M_1,

$$\epsilon_1 = \frac{3}{16\pi} \frac{1}{(\mathcal{Y}_N{}^\dagger \mathcal{Y}_N)_{11}} \sum_{n \neq 1} \mathrm{Im}\left[(\mathcal{Y}_N{}^\dagger \mathcal{Y}_N)^2_{1n} \right] \frac{M_1}{M_n}, \quad (7)$$

where $\mathcal{Y}_N$ is the matrix of Yukawa couplings in a weak basis while the right-handed Majorana mass matrix is diagonal. Such a reduction to ϵ_1 is possible since the asymmetry at M_3 was washed out by the process at M_2 and, subsequently, the asymmetry at M_2 was again washed out by the process at M_1 [10–12].

κ in (5) is determined by solving the full Boltzmann equations. κ can be simply parameterized in terms of K defined as the ratio of Γ_1, the decay width of $\nu_{\mathrm{R}1}$, to H, the Hubble parameter at temperature M_1 [7], where $K < 1$ describes processes out of thermal equilibrium and $\kappa < 1$ describes the wash-out effect,

$$-\kappa \simeq \frac{0.3}{K\,(\ln K)^{0.6}}, \quad (8)$$

for $10 \lesssim K = \Gamma_1/H \lesssim 10^6$, and

$$-\kappa \sim \frac{1}{2\sqrt{K^2+9}}, \quad (9)$$

for $0 \lesssim K \lesssim 10$ [7,13,14]. Even if the decay is much slower than the expansion rate of the universe, there is an unavoidable wash-out effect by $\kappa \sim 0.1$ [15], the size of which does not vary significantly as K varies 0 to 10. Thus, the lepton asymmetry in (5) requires a lower limit in CP asymmetry, say $\epsilon_1 \gtrsim 10^{-6}$ considering $Y_L \gtrsim 10^{-9}$.

It has been verified that the decay processes of heavy Majorana neutrinos can generate a sufficient amount of asymmetry and the asymmetry could be kept from wash-out in models with neutrinos based on the see-saw mechanism [9,14–17]. The see-saw mechanism [18] links the low-energy parameters (experimental observables) to the high-energy parameters (masses of the SM singlet neutrinos) through 15 independent parameters of the Yukawa matrix. Whichever the direction of the prediction by the see-saw mechanism goes, top-down or bottom-up, the 15 parameters make a full contribution in general to the prediction of any physical parameters, angles, CP phases, or mass eigenvalues at low energy or at high energy. So far, on the other hand, the lack in understanding of the Yukawa parameters has restricted consideration of the low–high link so as to rely on the choice of a model. For instance, there are models which predict low-energy observables in terms of the parameters of leptogenesis [19–22,17] and/or minimize the degree of freedom in parameter space of leptogenesis [23,24] at present.

The purpose of this paper is to find the structure of the matrix of the neutrino Yukawa couplings (Yukawa matrix) which satisfy three constraints simultaneously:

(i) the out-of-equilibrium condition of leptogenesis,
(ii) the upper bound of the lepton flavor violating decay in the supersymmetric standard model (MSSM),
(iii) the prediction of a large mixing angle in atmospheric neutrino oscillations [25] and the CHOOZ bound [26] of reactor neutrinos through the see-saw mechanism.

The structure of the Yukawa matrix is understood as an assembly of a matrix of eigenvalues, left transformation and right transformation. An advantage of the above point of view over mass structure, to say, in bi-unitary parametrization, compared with taking a whole mass structure in the weak basis, is that the coexistence of any large mixing angle and hierarchical mass spectrum can be clearly defined. Another advantage is that the application of constraints can be systematic because the constraint (i) is irrelevant to the left transformation while the constraint (ii) is irrelevant to the right transformation as we will see. While the constraints (i) and (ii) have separate dependences on the right mixing and the left mixing, respectively, the constraint (iii) is connected to both left mixing and right mixing matrices. A series of phenomenological constraints were carefully applied to the three parts of the Yukawa structure so as to partition parameter space into an eligible part and a ruled-out part.

First, the out-of-equilibrium condition of the lepton number-violating process constrains the right-handed transformation. In Sect. 2, an appropriate parametrization for a model comparison is introduced so that the size of CP asymmetry for leptogenesis can be written in a minimal way. In Sect. 3, the out-of-equilibrium condition is rephrased in an effective form fit to our purposes, and we examine it model by model depending on the leading contribution which gives rise to a comparison with the bound of the condition. Each model has its own characteristic range of elements of right transformation and the expression for the CP asymmetry from the leading contribution is determined in each model.

In Sect. 4, the strong bound of the branching ratio of $\mu \to e\gamma$ in MSSM is briefly reviewed. One can derive the constraint (ii) on the Yukawa couplings focused on left mixing angles from the renormalization group equation (RGE) of scalar mass terms. As for mixing angles, no pattern of left transformation is ruled out by constraint (ii). When the consideration is accompanied with a comparison between eigenvalues, a particular pattern can be ruled out by the constraint. The combination of two transformation matrices satisfied by (i) and (ii), with the type of eigenvalues, i.e., nearly degenerate or hierarchical, will be tested for whether they can predict a large mixing angle solution in atmospheric neutrinos and CHOOZ bound in reactor neutrinos through the see-saw mechanism. We have only seven characteristic combinations which satisfy the three constraints in Tables 2 and 3. The number of combinations is a consequence of excluding the cases where a suppressed eigenvalues can be derived only by fine tuning. There are four such cases shown in Appendix A.2. From Table 5 in Appendix A.2, we will see that the constraint (ii) is redundant if we first apply two other constraints, (i) and (iii).

In Sect. 5, we discuss the direction for the improvement of the models and the possible aspect of application of further constraints in a bi-unitary parametrization. A particular model will be examined being an appealing one for leptogenesis. A summary follows.

2 Bi-unitary transformation of neutrino Yukawa matrix

The leptogenesis has attracted attention recently because the see-saw mechanism is a plausible way for providing the scale for a lepton number-violating process in the early universe. The observed neutrino mass spectrum in experiments is consistent with a see-saw model with two scales, $\Lambda_{\rm EW}$, the electroweak (EW) symmetry breaking scale, and $\Lambda_{\rm GUT}$, the scale of grand unified theory (GUT). In the low-energy effective theory, a lepton number-violating interaction results in light neutrino masses by a non-zero vacuum expectation value of a Higgs scalar, and the mass matrix is diagonalized by three mixing angles, one Dirac phase and two Majorana phases. As far as the renormalizable gauge invariant couplings are concerned, the lepton sector will be

$$\mathcal{L} = -\mathcal{Y}_E H\bar{\ell}e_{\rm R} - \mathcal{Y}_N H\bar{\ell}\nu_{\rm R} - \frac{1}{2}M_{\rm R}\overline{\nu^c_{\rm R}}\nu_{\rm R} + \text{h.c.} \quad (10)$$

in the weak basis. The physical basis of light neutrinos and heavy neutrinos where the 6×6 mass matrix is diagonal can be obtained by a 6×6 unitary transformation from the weak basis of left-handed neutrinos in the ℓ and right neutrinos $\nu_{\rm R}$. This full transformation is reduced to the MNS [27] in the low-energy limit, in the basis where the charged lepton mass matrix and the right neutrino mass matrix $M_{\rm R}$ are diagonal.

Considering that the Yukawa matrix is not necessarily diagonal in the physical basis where the masses of the Majorana neutrinos are diagonal, the transformation of the Yukawa matrix (neutrino Dirac mass matrix) needs to be defined in addition to the transformation of the Majorana mass. However, there might exist constraints that can connect the MNS to the left-handed mixing matrix of the Yukawa matrix, for example, arising from a flavor symmetry. Choose a basis where $M_{\rm R} = \text{Diag}\,(M_1, M_2, M_3)$. Let

$$\mathcal{Y}_D = \text{Diag}\,(y_1, y_2, y_3)\,,$$

so that a 3×3 matrix $\mathcal{Y}_N$ for three-generation neutrinos can be expressed using the bi-unitary transformation as follows:

$$\mathcal{Y}_N = \mathbb{L}\mathcal{Y}_D\mathbb{R}^\dagger. \quad (11)$$

Pascoli, Petcov, and Rodejohann also discussed this parametrization in [28] in comparison with other parameterizations to describe leptogenesis. The unitarity of $\mathbb{L}$ implies that no left-handed mixing angle in the Yukawa matrix affects any parameters in leptogenesis, because the loop contribution appears in the terms $\mathcal{Y}_N^\dagger\mathcal{Y}_N = \mathbb{R}\mathcal{Y}_D^2\mathbb{R}^\dagger$. We have

$$\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{11} = y_1^2|R_{11}|^2 + y_2^2|R_{12}|^2 + y_3^2|R_{13}|^2, \quad (12)$$

$$\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{1k} = y_1^2 R_{11}R_{k1}^* + y_2^2 R_{12}R_{k2}^* + y_3^2 R_{13}R_{k3}^*, \quad (13)$$

where R_{ij} is an element of $\mathbb{R}$ and $k = 2, 3$. Expressing the CP asymmetry in (6) in terms of the Yukawa matrices yields

$$\epsilon_1 \approx 10^{-1}\text{Im}\left[\frac{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{12}^2}{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{11}}\frac{M_1}{M_2} + \frac{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{13}^2}{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{11}}\frac{M_1}{M_3}\right], \quad (14)$$

and all the possible models can then be classified into three cases depending on the dominant term in (12). Factors will be indicated in bold strokes in the following if they are much smaller than order 1. These lead to the following cases. However, the terms with the bold factors cannot be necessarily neglected in the leptogenesis analysis.

Case I

Case I, where $y_3^2|R_{13}|^2$ is dominant,

$$\begin{aligned}\frac{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{1k}^2}{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{11}} &\quad (15)\\ \approx \left(\frac{y_1^2R_{11}R_{k1}^*}{y_3|R_{13}|} + \frac{y_2^2R_{12}R_{k2}^*}{y_3|R_{13}|} + \frac{y_3^2R_{13}R_{k3}^*}{y_3|R_{13}|}\right)^2 &\\ = \left(\frac{\mathbf{y_1|R_{11}|}}{\mathbf{y_3|R_{13}|}}y_1R_{k1}^*\text{e}^{\text{i}\gamma_1} + \frac{\mathbf{y_2|R_{12}|}}{\mathbf{y_3|R_{13}|}}y_2R_{k2}^*\text{e}^{\text{i}\gamma_2} + y_3R_{k3}^*\text{e}^{\text{i}\gamma_3}\right)^2, &\end{aligned}$$

where γ_1, γ_2, and γ_3 are arguments of R_{11}, R_{12}, and R_{13}, respectively.

Case II

Case II, where $y_2^2|R_{12}|^2$ is dominant,

$$\begin{aligned}\frac{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{1k}^2}{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{11}} &\quad (16)\\ \approx \left(\frac{y_1^2R_{11}R_{k1}^*}{y_2|R_{12}|} + \frac{y_2^2R_{12}R_{k2}^*}{y_2|R_{12}|} + \frac{y_3^2R_{13}R_{k3}^*}{y_2|R_{12}|}\right)^2 &\\ = \left(\frac{\mathbf{y_1|R_{11}|}}{\mathbf{y_2|R_{12}|}}y_1R_{k1}^*\text{e}^{\text{i}\gamma_1} + y_2R_{k2}^*\text{e}^{\text{i}\gamma_2} + \frac{\mathbf{y_3|R_{13}|}}{\mathbf{y_2|R_{12}|}}y_3R_{k3}^*\text{e}^{\text{i}\gamma_3}\right)^2, &\end{aligned}$$

Case III

Case III, where $y_1^2|R_{11}|^2$ is dominant,

$$\frac{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{1k}^2}{\left(\mathcal{Y}_N^\dagger\mathcal{Y}_N\right)_{11}} \approx \left(\frac{y_1^2R_{11}R_{k1}^*}{y_1|R_{11}|} + \frac{y_2^2R_{12}R_{k2}^*}{y_1|R_{11}|} + \frac{y_3^2R_{13}R_{k3}^*}{y_1|R_{11}|}\right)^2$$

$$= \left(y_1 R^*_{k1} \mathrm{e}^{\mathrm{i}\gamma_1} + \frac{\mathbf{y_2|R_{12}|}}{\mathbf{y_1|R_{11}|}} y_2 R^*_{k2} \mathrm{e}^{\mathrm{i}\gamma_2} + \frac{\mathbf{y_3|R_{13}|}}{\mathbf{y_1|R_{11}|}} y_3 R^*_{k3} \mathrm{e}^{\mathrm{i}\gamma_3} \right)^2 , \quad (17)$$

Case IV

Case IV, where $y_1^2|R_{11}|^2$, $y_2^2|R_{12}|^2$, and $y_3^2|R_{13}|^2$ are equally dominant,

$$\frac{\left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{1k}^2}{\left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11}} \approx \left(\frac{y_1^2 R_{11} R^*_{k1}}{y_1|R_{11}|} + \frac{y_2^2 R_{12} R^*_{k2}}{y_2|R_{12}|} + \frac{y_3^2 R_{13} R^*_{k3}}{y_3|R_{13}|} \right)^2 = \left(y_1 R^*_{k1} \mathrm{e}^{\mathrm{i}\gamma_1} + y_2 R^*_{k2} \mathrm{e}^{\mathrm{i}\gamma_2} + y_3 R^*_{k3} \mathrm{e}^{\mathrm{i}\gamma_3} \right)^2 . \quad (18)$$

There are also cases with two equally dominant terms in $\left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11}$ in (12).

Case II*

Case II*, where $y_3^2|R_{13}|^2, y_2^2|R_{12}|^2 \gg y_1^2|R_{11}|^2$.

Case III*

Case III*, where $y_3^2|R_{13}|^2, y_1^2|R_{11}|^2 \gg y_2^2|R_{12}|^2$.

Case IV*

Case IV*, where $y_1^2|R_{11}|^2, y_2^2|R_{12}|^2 \gg y_3^2|R_{13}|^2$.

The expression corresponding to

$$\left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{1k}^2 \Big/ \left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11}$$

can be found with a single suppressed factor. Such three possible cases have an aspect similar to the cases II, III, and IV when the out-of-equilibrium condition of our lepton number-violating process constrains the magnitude of $\mathbb{R}$, as is to be discussed further in the next section.

The hermitian operator $\mathcal{Y}_N^\dagger \mathcal{Y}_N$ can be expressed with three independent CP phases and six real parameters, such as

$$\mathcal{Y}_N^\dagger \mathcal{Y}_N = \begin{pmatrix} z_{11} & z_{12}\mathrm{e}^{\mathrm{i}\phi_1} & z_{13}\mathrm{e}^{\mathrm{i}\phi_2} \\ z_{12}\mathrm{e}^{-\mathrm{i}\phi_1} & z_{22} & z_{23}\mathrm{e}^{\mathrm{i}\phi_3} \\ z_{13}\mathrm{e}^{-\mathrm{i}\phi_2} & z_{23}\mathrm{e}^{-\mathrm{i}\phi_3} & z_{33} \end{pmatrix} , \quad (19)$$

where the z_{ij} are real. Two phases, say ϕ_1 and ϕ_2, can be eliminated by a diagonal phase transformation, $\mathbb{P} \equiv \mathrm{Diag}\,(1, \exp(-\mathrm{i}\phi_1), \exp(-\mathrm{i}\phi_2))$. We have

$$\mathbb{P}^\dagger \mathcal{Y}_N^\dagger \mathcal{Y}_N \mathbb{P} = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23}\mathrm{e}^{\mathrm{i}\delta'} \\ z_{13} & z_{23}\mathrm{e}^{-\mathrm{i}\delta'} & z_{33} \end{pmatrix} , \quad (20)$$

where $\delta' \equiv -\phi_1 - \phi_2 + \phi_3$. In general, the diagonalization of $\mathbb{P}\mathcal{Y}_N^\dagger \mathcal{Y}_N \mathbb{P}^\dagger$ still requires three independent phases as a combination of two Majorana phases and one Dirac phase. One can take the absolute values of the elements of $\mathbb{R}$ in (15)–(17) to find $|\mathcal{Y}_N{}^\dagger \mathcal{Y}_N|^2_{1k}$, regardless of the complexity with γ_1, γ_2, and γ_3 in (15)–(17) in terms of elements of $\mathbb{R}$. For example, the size of the CP asymmetry in (14) can be written as

$$\epsilon_1 \approx 10^{-1} \frac{|\mathcal{Y}_N^\dagger \mathcal{Y}_N|_{12}^2}{\mathcal{Y}_N^\dagger \mathcal{Y}_{N\,11}} \sin 2\phi_1 \frac{M_1}{M_2} + 10^{-1} \frac{|\mathcal{Y}_N^\dagger \mathcal{Y}_N|_{13}^2}{\mathcal{Y}_N^\dagger \mathcal{Y}_{N\,11}} \sin 2\phi_2 \frac{M_1}{M_3}, \quad (21)$$

using (19).

3 Out-of-equilibrium condition and its constraints on Yukawa mixing angles

The decay width of $\nu_{\mathrm{R}1}$ by the Yukawa interaction at tree level is

$$\Gamma_1 = \frac{1}{8\pi} \left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11} M_1 . \quad (22)$$

The out-of-equilibrium condition results when the Hubble parameter exceeds the decay rate, expressing the Hubble parameter in terms of temperature T,

$$\Gamma_1 < H = 1.66 g_*^{1/2} \frac{T^2}{M_{\mathrm{Pl}}} . \quad (23)$$

At temperature $T = M_1$, the condition can be rephrased as

$$\left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11} \lesssim \zeta_1^2 , \quad (24)$$

where

$$\zeta_1^2 \equiv 10^2 \frac{M_1}{M_{\mathrm{Pl}}} . \quad (25)$$

The upper bound ζ_1^2 in (24) cannot exceed M_1/M_{GUT}, when M_{GUT} is chosen in the range $(10^{-4}$–$10^{-3})M_{\mathrm{Pl}}$ depending on the theoretical framework, and the masses of heavy Majorana neutrinos cannot be higher than M_{GUT}. As long as M_1 is hierarchically smaller than the other masses of the heavy neutrinos, as assumed here, the ζ_1^2 should be considered safely as small as of the order of $\lesssim 10^{-2}$.

The condition was examined in detail by Buchmuller and Plumacher [12] with the relation of the generated $B-L$ asymmetry to the effective mass defined as

$$\tilde{m}_1 = \left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11} \frac{v_2^2}{M_1} . \quad (26)$$

Besides the upper bound which keeps the asymmetry protected from the wash-out effect, there is also a lower bound for $\tilde{m}_1$ since, at high temperature, weak Yukawa couplings cannot produce enough neutrinos. The eligible $\tilde{m}_1$ capable of generating sufficient $B-L$ (or equivalently L) asymmetry, lies in the range $m_1 < \tilde{m}_1 < m_3$ due to the see-saw mechanism [17,21], where m_1 is the mass of the lightest neutrino, which is presumed to be non-zero but smaller than $m_2 \approx \sqrt{\Delta m^2_\odot}$ and m_3 is the mass of the heaviest light neutrino, whose magnitude is about $\sqrt{\Delta m^2_{\rm atm}}$, assuming a hierarchical light neutrino spectrum.

Using (12), the condition in (24) can be rephrased in a convenient way as

$$y_1^2|R_{11}|^2 + y_2^2|R_{12}|^2 + y_3^2|R_{13}|^2 \lesssim \zeta_1^2. \tag{27}$$

The first significant implication of the above condition is

$$|R_{13}| \lesssim \zeta_1, \tag{28}$$

when y_3 is of order 1. The hierarchy in the masses of right-handed neutrinos is a preliminary for the CP asymmetry in (7) of leptogenesis, causing the smallness of ζ_1 and so a small-angle constraint for $|R_{13}|$. The next implication is the smallness of y_1,

$$y_1 \lesssim \zeta_1, \tag{29}$$

since $|R_{11}|$ is always of order 1 whether mixing angles are large or small. So y_1 cannot be of the same approximate size as y_3 if $y_3 \sim 1$. The models that satisfy the two conditions in (28)–(29) can be considered according to which term dominates in the real number $(\mathcal{Y}_N^\dagger \mathcal{Y}_N)_{11}$ and how different the eigenvalues y_1, y_2, and y_3 of the Yukawa matrix are. A model with $y_3 \gg y_2 \gg y_1$ is denoted Case (a), the one with $y_3 \sim y_2 \gg y_1$ by Case (b), and the one with $y_3 \gg y_2 \sim y_1$ by Case (c). The leading contribution to ϵ_1 in Table 1 is represented by the first term in (14).

In Case I in (15), if the eigenvalues are of normal hierarchy, $y_3 \gg y_2$, the out-of-equilibrium condition in (27) does not place any constraints on the elements in $\mathbb{R}$ other than $|R_{13}| < \zeta_1$, while the nearly degenerate eigenvalues, $y_3 \sim y_2$, restrict $|R_{12}|$ to be small. If the transformation $\mathbb{R}/\mathbb{J} \equiv \mathbb{R}\mathbb{J}^{-1}$ is parameterized as

$$\mathbb{R}/\mathbb{J} = \begin{pmatrix} c_{13}c_{12} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - s_{23}s_{13}c_{12}\mathrm{e}^{-\mathrm{i}\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}\mathrm{e}^{-\mathrm{i}\delta} & s_{23}c_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ s_{23}s_{12}\mathrm{e}^{\mathrm{i}\delta} - s_{13}c_{23}c_{12} & -s_{23}c_{12}\mathrm{e}^{\mathrm{i}\delta} - s_{13}s_{12}c_{23} & c_{23}c_{13} \end{pmatrix}, \tag{30}$$

where $\mathbb{J}$ is a diagonal phase transformation, one can deduce the smallness of $|R_{21}|$ and $|R_{31}|$ from the smallness of R_{12} due to the small s_{12}. The possible structures of $\mathbb{R}$ for Case I are listed in Table 1, where the s_{12} and s_{23} specified in the table are allowed to be either a small angle or a large one, while $\lambda_{\rm s}$ indicates that only a small angle is allowed. If s_{23} is not so small, the leading contribution of the imaginary part which leads to ϵ_1 mainly consists of $(y_3|R_{23}|)^2$ and M_1/M_2. The model examined in [17] is an example of Case I(a).

In Case II, in (16), if the eigenvalues are of normal hierarchy with $y_3 \gg y_2$, the out-of-equilibrium condition allows $|R_{12}|$ to be much larger than $|R_{13}|$. If the assumption is made that y_2 is not larger than ζ_1, then $|R_{12}|$ can still be a large angle. In other words, $|R_{12}|$ can be either small or large, though it is larger than ζ_1 in this model. The leading contribution to ϵ_1 depends mainly on

$$y_2|R_{22}| + \frac{\mathbf{y_3|R_{13}|}}{\mathbf{y_2|R_{12}|}} y_3|R_{23}|. \tag{31}$$

Otherwise, the two eigenvalues, y_2 and y_3 in Case II, are close to each other. The element $|R_{12}|$ is smaller than ζ_1, and $|R_{13}|$ is far smaller than $|R_{12}|$. The term

$$|\mathcal{Y}_N^\dagger \mathcal{Y}_N|_{12}{}^2 / \left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11}$$

reduces to $y_2|R_{22}|$ which is now the leading contribution to ϵ_1 of order 1. The assumption implies $|R_{12}| \gg y_1/y_2$, so that the Case II(c) with $y_1 \sim y_2$ is ruled out. For the same reason, there cannot occur Case II*(c), while Case II*(a) with $y_3 \gg y_2$ and Case II*(b) with $y_3 \sim y_2$ and small $|R_{12}|$ are allowed. The allowed ranges in the elements of $|\mathbb{R}|$ for Case II*(a) and Case II*(b) can be found to be the same as in the Case II(a) and Case II(b), respectively.

Case III in (17) corresponds to $|R_{12}| \ll y_1/y_2$, i.e., $|R_{12}|$ is always a small angle. No part of the contribution to $|\mathcal{Y}_N^\dagger \mathcal{Y}_N|_{12}{}^2 / \left(\mathcal{Y}_N^\dagger \mathcal{Y}_N\right)_{11}$ is yet ruled out by the assumptions given here. Case III* gives rise to a similar constraint on $|R_{12}|$ as Case III does.

In Case IV in (18), we have a subcase with $y_3 \sim y_2$, and $|R_{12}|$ is not allowed by the out-of-equilibrium condition because $y_2|R_{12}|$ should not be larger than ζ_1. This is the case with $|R_{12}| \sim y_1/y_2$. Thus, $|R_{12}|$ is small for $y_2 \gg y_1$, while $|R_{12}|$ is large for $y_2 \sim y_1$. The possible structures of $|\mathbb{R}|$ derived in Case IV* also is consistent with those in Case IV.

The structures of $|\mathbb{R}|$ in Cases II*, III*, IV* attribute their correspondences to those in Cases II, III, IV, respectively; they have relative significance in comparison to $y_1|R_{11}|$ and $y_2|R_{12}|$ when $y_3|R_{13}|$ is fixed. In other words, once $|R_{13}|$ is bounded, cases can be effectively classified into a case with $y_1|R_{11}| \ll y_2|R_{12}|$, a case with $y_2|R_{12}| \ll y_1|R_{11}|$, and a case with $y_1|R_{11}| \sim y_2|R_{12}|$. Thus, the cases with two equal dominant terms can remain without further detailed discussion.

In Table 1, where the various cases are considered, the allowed structures of $|\mathbb{R}|$ under those cases and the CP asymmetry ϵ_1 in terms of the leading contribution after ruling out the suppressed part under the assumptions are summarized. In Case I and Case IV with $|R_{23}|$ of order 1, or in Case II(b), ϵ_1 can reach its model independent maximum value,

$$\epsilon_{1\max} \sim 10^{-1}\frac{M_1}{M_2}, \tag{32}$$

where $M_1 < M_2$.

Table 1. Model classification: Cases I, II, III, and IV are introduced in (15)–(18). Case (a) is for $y_3 \gg y_2 \gg y_1$, Case (b) for $y_3 \sim y_2 \gg y_1$, and Case (c) for $y_3 \gg y_2 \sim y_1$. In $|\mathbb{R}|$, ζ_1 is defined in (24), the $\zeta_{\rm s}$ in Cases I(b), IV(b) and in Case II(b) represent small values of $|R_{13}|$ and $|R_{12}|$, respectively, while $\zeta_{\rm s}$ in Case III and Case IV(a) represent simply a small angle. The s_{12} and s_{23} imply that those angles can be large or small. The Case II and the Case (c) cannot be compatible

	Models	$\lvert\mathbb{R}\rvert \sim$	ϵ_1 in terms of leading contribution
I(a),(c)	$\begin{cases} y_3^2\lvert R_{13}\rvert^2 \gg y_2^2\lvert R_{12}\rvert^2, y_1^2\lvert R_{11}\rvert^2 \\ y_3 \gg y_2 \gg y_1 \text{ or } y_3 \gg y_2 \sim y_1 \end{cases}$	$\begin{bmatrix} 1 & s_{12} & <\zeta_1 \\ s_{12} & 1 & s_{23} \\ s_{12}s_{23} & s_{23} & 1 \end{bmatrix}$	$10^{-1}\left(y_3\lvert R_{23}\rvert\right)^2 \frac{M_1}{M_2}$
I(b)	$\begin{cases} y_3^2\lvert R_{13}\rvert^2 \gg y_2^2\lvert R_{12}\rvert^2, y_1^2\lvert R_{11}\rvert^2 \\ y_3 \sim y_2 \gg y_1 \end{cases}$	$\begin{bmatrix} 1 & \ll\zeta_{\rm s} & <\zeta_1 \\ \ll\zeta_{\rm s} & 1 & s_{23} \\ \ll\zeta_{\rm s} & s_{23} & 1 \end{bmatrix}$	$10^{-1}\left(y_3\lvert R_{23}\rvert\right)^2 \frac{M_1}{M_2}$
II(a)	$\begin{cases} y_2^2\lvert R_{12}\rvert^2 \gg y_3^2\lvert R_{13}\rvert^2, y_1^2\lvert R_{11}\rvert^2 \\ y_3 \gg y_2 \gg y_1 \end{cases}$	$\begin{bmatrix} 1 & s_{12} & <\zeta_1 \\ s_{12} & 1 & s_{23} \\ s_{12}s_{23} & s_{23} & 1 \end{bmatrix}$	$10^{-1}\left\{y_2\lvert R_{22}\rvert + \frac{\mathbf{y_3\lvert R_{13}\rvert}}{\mathbf{y_2\lvert R_{12}\rvert}} y_3\lvert R_{23}\rvert\right\}^2 \frac{M_1}{M_2}$
II(b)	$\begin{cases} y_2^2\lvert R_{12}\rvert^2 \gg y_3^2\lvert R_{13}\rvert^2, y_1^2\lvert R_{11}\rvert^2 \\ y_3 \sim y_2 \gg y_1 \end{cases}$	$\begin{bmatrix} 1 & <\zeta_1 & \ll\zeta_{\rm s} \\ <\zeta_1 & 1 & s_{23} \\ <\zeta_1 & s_{23} & 1 \end{bmatrix}$	$10^{-1}\left(y_2\lvert R_{22}\rvert\right)^2 \frac{M_1}{M_2}$
III(a),(b),(c)	$\begin{cases} y_1^2\lvert R_{11}\rvert^2 \gg y_2^2\lvert R_{12}\rvert^2, y_3^2\lvert R_{13}\rvert^2 \\ y_3 \gg y_2 \gg y_1, y_3 \sim y_2 \gg y_1, \\ \text{or } y_3 \gg y_2 \gg y_1 \end{cases}$	$\begin{bmatrix} 1 & \zeta_{\rm s} & <\zeta_1 \\ \zeta_{\rm s} & 1 & s_{23} \\ \zeta_{\rm s} & s_{23} & 1 \end{bmatrix}$	$10^{-1}\begin{Bmatrix} y_1\lvert R_{21}\rvert + \frac{\mathbf{y_2\lvert R_{12}\rvert}}{\mathbf{y_1\lvert R_{11}\rvert}} y_2\lvert R_{22}\rvert \\ + \frac{\mathbf{y_3\lvert R_{13}\rvert}}{\mathbf{y_1\lvert R_{11}\rvert}} y_3\lvert R_{23}\rvert \end{Bmatrix}^2 \frac{M_1}{M_2}$
IV(a),(b)	$\begin{cases} y_3^2\lvert R_{13}\rvert^2 \sim y_2^2\lvert R_{12}\rvert^2 \sim y_1^2\lvert R_{11}\rvert^2 \\ y_3 \gg y_2 \gg y_1 \text{ or } y_3 \sim y_2 \gg y_1 \end{cases}$	$\begin{bmatrix} 1 & \zeta_{\rm s} & <\zeta_1 \\ \zeta_{\rm s} & 1 & s_{23} \\ \zeta_{\rm s} & s_{23} & 1 \end{bmatrix}$	$10^{-1}\{y_2\lvert R_{22}\rvert + y_3\lvert R_{23}\rvert\}^2 \frac{M_1}{M_2}$
IV(c)	$\begin{cases} y_3^2\lvert R_{13}\rvert^2 \sim y_2^2\lvert R_{12}\rvert^2 \sim y_1^2\lvert R_{11}\rvert^2 \\ y_3 \gg y_2 \sim y_1 \end{cases}$	$\begin{bmatrix} 1 & 1 & <\zeta_1 \\ 1 & 1 & s_{23} \\ s_{23} & s_{23} & 1 \end{bmatrix}$	$10^{-1}\{y_2\lvert R_{22}\rvert + y_3\lvert R_{23}\rvert\}^2 \frac{M_1}{M_2}$

4 Low-energy observables and their constraints on Yukawa mixing transformations

The current experimental limit on the branching ratio (Br) of the lepton flavor violating (LFV) decay mode, $\mu \to e\gamma$, is

$$\mathrm{Br}\left(\mu \to e\gamma\right) < 1.2 \times 10^{-11}. \tag{33}$$

In the SM, the branching ratio is $< 10^{-50}$ and is suppressed far below the observable bound [29]. Another theoretical framework where one can estimate the size of the LFV decay rates is the minimal supersymmetry (SUSY) standard model (MSSM), where the rates for LFV processes can be enhanced due to a large $\tan\beta$ [30]. The neutrino Yukawa couplings cause the renormalization group equations (RGE) to develop a flavor changing contribution to soft masses of scalar leptons, which may originate from a universal value for all kinds of scalar leptons at the GUT scale M_X. The generated LFV mass terms for scalar leptons after integrating RGE are

$$\begin{aligned} \left(\Delta m_{\tilde{L}}^2\right)_{ij} &\approx \frac{-1}{8\pi^2}\left(3m_0^2 + A_0^2\right)\sum_k \mathcal{Y}_{Nik} \log\frac{M_X}{M_k}\mathcal{Y}_{Nkj}^\dagger \\ &\sim \frac{-1}{8\pi^2}\left(3m_0^2 + A_0^2\right)\log\frac{M_X}{M_3}\left(\mathcal{Y}_N\mathcal{Y}_N^\dagger\right)_{ij}, \end{aligned} \tag{34}$$

where $\log\frac{M_X}{M_k}$ for $k = 1$–3 can be considered to be of the same order even when the M_k are in agreement with the hierarchy. The m_0^2 and A_0^2 are the universal masses and universal trilinear couplings, respectively, in the soft SUSY-breaking lagrangian [30]. The flavor changing decay in (33) involves neutralino exchange and chargino exchange in loop diagrams, and the resulting amplitude is proportional to $\tan\beta$. Therefore the branching ratios can be expressed in terms of $\mathcal{Y}_N\mathcal{Y}_N^\dagger$ as follows:

$$\mathrm{Br}\left(\ell_i \to \ell_j\gamma\right) \tag{35}$$

$$\sim \frac{\alpha^3}{G_F^2 m_s^8}\left|\frac{-1}{8\pi^2}\left(3m_0^2+A_0^2\right)\log\frac{M_X}{M_3}\left(\mathcal{Y}_N\mathcal{Y}_N^\dagger\right)_{ij}\right|^2 \times \tan^2\beta,$$

where m_s is the typical mass of a superparticle, α is the fine structure constant, and G_F is the Fermi constant [30, 31]. Recently, LFV in low energy has been considered for its relevance for leptogenesis [24,32].

Even if a model can predict an enhanced branching ratio near the range accessible in a near future experiment, the eligible structure of the Yukawa matrix implies that there might be a constraint for severely suppressed mixing angles in a left-handed transformation of the Yukawa matrix at low energy. The enhancement mechanism of the branching ratios is possible not only by increasing $\tan\beta$ but also by increasing the mixing angles. Hisano et al. [30] analyzed the CKM matrix of quark mixing. With small mixing angles as in CKM, the branching ratios of the $\ell_i \to \ell_j\gamma$ processes for large $\tan\beta$ can come close to the current experimental bounds in (33). Considering the strong upper bound in the current experimental limit for the decay mode $\mu \to e\gamma$, the relevant constraint on the Yukawa couplings is, in terms of the elements of $\mathbb{L}$ defined in (11),

$$\left(\mathcal{Y}_N\mathcal{Y}_N^\dagger\right)_{21} = y_1^2 L_{21}L_{11}^* + y_2^2 L_{22}L_{12}^* + y_3^2 L_{23}L_{13}^* \ll 1, \quad (36)$$

which can be satisfied only if y_2 or $|L_{12}|$ is small and $|L_{23}|$ or $|L_{13}|$ is small.

In the rest of this section, I combine the constraint on $\mathbb{R}$ from the out-of-equilibrium condition and the constraint on $\mathbb{L}$ from the upper bound of the branching ratio of $\mu \to e\gamma$. The question is whether the result of those combinations can be accommodated with the experimentally observed large mixing angles of light neutrinos when implemented via the see-saw mechanism. Light neutrino masses can be obtained through the see-saw mechanism where Dirac masses are defined in terms of the non-zero vacuum expectation value v_2 of a light Higgs in MSSM with two Higgs doublets,

$$\begin{aligned} m_\nu &= -v_2^2 \mathcal{Y}_N M_R^{-1}\mathcal{Y}_N^T \\ &= -v_2^2 \mathbb{L}\mathcal{Y}_D\mathbb{R}^\dagger M_R^{-1}\mathbb{R}^*\mathcal{Y}_D^T\mathbb{L}^T. \end{aligned} \quad (37)$$

In this basis the mass matrix of the charged leptons is diagonal. When the transformations $\mathbb{L}$ and $\mathbb{R}$ are trivial, i.e., equal to the identity, the see-saw mechanism yields the light neutrino mass matrix,

$$m_\nu = -v_2^2 \mathrm{Diag}\left(\frac{y_1^2}{M_1}, \frac{y_2^2}{M_2}, \frac{y_3^2}{M_3}\right), \quad (38)$$

from (37). It will be interesting to watch how the neutrino mass matrix in (37) can change as the type of transformation matrix $\mathbb{R}$, and subsequently $\mathbb{L}$, switches from one to another general form. There are four types of $\mathbb{R}$, which are given in Table 1, and the four types of $\mathbb{L}$ are as follows:

$$|\mathbb{R}_1| \sim \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & 1 \\ \rho & 1 & 1 \end{pmatrix}, \quad |\mathbb{L}_1| \sim \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix},$$

$$|\mathbb{R}_2| \sim \begin{pmatrix} 1 & 1 & \rho \\ 1 & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \quad |\mathbb{L}_2| \sim \begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix}, \quad (39)$$

$$|\mathbb{R}_3| \sim \begin{pmatrix} 1 & 1 & \rho \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad |\mathbb{L}_3| \sim \begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$|\mathbb{R}_4| \sim \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \quad |\mathbb{L}_4| \sim \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix},$$

which satisfy the out-of-equilibrium condition in (24). The most simplified notation is used, which is that diagonal elements and large mixing elements are denoted by 1, whereas small mixing elements are denoted by the arbitrarily small values having $\rho \ll 1$ and $\lambda \ll 1$ in $\mathbb{R}$ and $\mathbb{L}$, respectively. That is, there are only two kinds of elements in a transformation matrix: of order 1 or of order far less than 1. The more detailed $\mathbb{R}$ have been sorted in Table 1 by considering the out-of-equilibrium condition. Later, in case the leading order in an entry of light neutrino mass matrix originates from small angles in the transformation matrices, the light neutrino mass matrices in terms of the small angles, considering the actual entries as expansions of small angles, are shown for the models in Appendix A.1.

Each structure of $|\mathbb{L}_1|$–$|\mathbb{L}_4|$ is similar to each of $|\mathbb{R}_1|$–$|\mathbb{R}_4|$, as shown in (39), which satisfy the condition in (36) which is of the same form as (27) for the $\mathbb{R}$ matrix. For the suppressed $(\mathcal{Y}_N\mathcal{Y}_N^\dagger)_{21}$, it is necessary that L_{12} and $L_{13}L_{23}$ be small for $y_2 \sim y_3$, while the smallness of L_{12} or L_{23} is optional for $y_2 \ll y_3$ and L_{13} small. The possibility of large $|L_{13}|$ is excluded by the CHOOZ bound [26] on the MNS neutrino mixing matrix.

In Tables 2 and 3, all the possible structures of the light neutrino mass matrix, which can be produced by (37) using the $\mathbb{R}$ and $\mathbb{L}$ in (39), are listed. Due to $M_1 \ll M_2, M_3$ being assumed for (7), the following approximation can be used in the tables:

$$\begin{aligned} &M_{123}^{-1} \sim M_{12}^{-1} \sim M_1^{-1}, \\ &M_{123}^{-1} \gg M_{23}^{-1}, \ M_{123}^{-1} \gg M_3^{-1}, \end{aligned} \quad (40)$$

where

$$\begin{aligned} &M_{123}^{-1} \equiv M_1^{-1} + M_2^{-1} + M_3^{-1}, \\ &M_{12}^{-1} \equiv M_1^{-1} + M_2^{-1}, \end{aligned} \quad (41)$$

etc., and

$$\begin{aligned} &y_{123} = y_1 + y_2 + y_3, \\ &y_{12} = y_1 + y_2, \ y_{23} = y_2 + y_3. \end{aligned} \quad (42)$$

The comparison between M_2 and M_3 is not yet fixed. Either possibility, $M_2 \ll M_3$ or $M_2 \sim M_3$, can still be considered.

Table 2. List of possible structures of the matrix m_ν/v_2^2 which can be produced by the see-saw mechanism in (37) using the $\mathbb{R}$ and $\mathbb{L}$ in (39), where m_ν is a matrix mass of light neutrinos and v_2 is the vacuum expectation value in two-Higgs-doublet models. M_{123} and M_{12} are defined in (41) and y_{123} and y_{12} are defined in (42). Please see Appendix A.1 for the entries marked by $*$

	$\mathbb{L}_1$	$\mathbb{L}_2$
$\mathbb{R}_1$	$\begin{bmatrix} y_1^2 M_1^{-1} & * & * \\ * & y_{23}^2 M_{23}^{-1} & y_{23}^2 M_{23}^{-1} \\ * & y_{23}^2 M_{23}^{-1} & y_{23}^2 M_{23}^{-1} \end{bmatrix}^a$	$\begin{bmatrix} y_1^2 M_1^{-1} + y_2^2 M_{23}^{-1} & y_1^2 M_1^{-1} + y_2^2 M_{23}^{-1} & y_2 y_3 M_{23}^{-1} \\ y_1^2 M_1^{-1} + y_2^2 M_{23}^{-1} & y_1^2 M_1^{-1} + y_2^2 M_{23}^{-1} & y_2 y_3 M_{23}^{-1} \\ y_2 y_3 M_{23}^{-1} & y_2 y_3 M_{23}^{-1} & y_3^2 M_{23}^{-1} \end{bmatrix}$
$\mathbb{R}_2$	$\begin{bmatrix} y_1^2 M_{12}^{-1} & y_1 y_2 M_{12}^{-1} & y_1 y_2 M_{12}^{-1} \\ y_1 y_2 M_{12}^{-1} & y_2^2 M_{12}^{-1} + y_3^2 M_3^{-1} & y_2^2 M_{12}^{-1} + y_3^2 M_3^{-1} \\ y_1 y_2 M_{12}^{-1} & y_2^2 M_{12}^{-1} + y_3^2 M_3^{-1} & y_2^2 M_{12}^{-1} + y_3^2 M_3^{-1} \end{bmatrix}^b$	$\begin{bmatrix} y_{12}^2 M_{12}^{-1} & y_{12}^2 M_{12}^{-1} & * \\ y_{12}^2 M_{12}^{-1} & y_{12}^2 M_{12}^{-1} & * \\ * & * & y_3^2 M_3^{-1} \end{bmatrix}$
$\mathbb{R}_3$	$\begin{bmatrix} y_1^2 M_{123}^{-1} & y_1 y_2 M_{123}^{-1} + y_1 y_3 M_{23}^{-1} & y_1 y_2 M_{123}^{-1} + y_1 y_3 M_{23}^{-1} \\ y_1 y_2 M_{123}^{-1} + y_1 y_3 M_{23}^{-1} & y_2^2 M_{123}^{-1} + y_{23} y_3 M_{23}^{-1} & y_2^2 M_{123}^{-1} + y_{23} y_3 M_{23}^{-1} \\ y_1 y_2 M_{123}^{-1} + y_1 y_3 M_{23}^{-1} & y_2^2 M_{123}^{-1} + y_{23} y_3 M_{23}^{-1} & y_2^2 M_{123}^{-1} + y_{23} y_3 M_{23}^{-1} \end{bmatrix}$	$\begin{bmatrix} y_{12}^2 M_{123}^{-1} & y_{12}^2 M_{123}^{-1} & y_{12} y_3 M_{23}^{-1} \\ y_{12}^2 M_{123}^{-1} & y_{12}^2 M_{123}^{-1} & y_{12} y_3 M_{23}^{-1} \\ y_{12} y_3 M_{23}^{-1} & y_{12} y_3 M_{23}^{-1} & y_3^2 M_{23}^{-1} \end{bmatrix}$
$\mathbb{R}_4$	$\begin{bmatrix} y_1^2 M_1^{-1} & * & * \\ * & y_2^2 M_2^{-1} + y_3^2 M_3^{-1} & y_2^2 M_2^{-1} + y_3^2 M_3^{-1} \\ * & y_2^2 M_2^{-1} + y_3^2 M_3^{-1} & y_2^2 M_2^{-1} + y_3^2 M_3^{-1} \end{bmatrix}^c$	$\begin{bmatrix} y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & * \\ y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & * \\ * & * & y_3^2 M_3^{-1} \end{bmatrix}$

[a] Model A: $(\mathbb{R}_1, \mathbb{L}_1)$, $y_2 \sim y_3$
[b] Model B: $(\mathbb{R}_2, \mathbb{L}_1)$, $y_2 \ll y_3$
[c] Model C: $(\mathbb{R}_4, \mathbb{L}_1)$, $y_2 \sim y_3$ Model D: $(\mathbb{R}_4, \mathbb{L}_1)$, $y_2 \ll y_3$

Table 3. The matrix m_ν/v_2^2 which can be produced by the see-saw mechanism in (37) using the $\mathbb{R}$ and $\mathbb{L}$ in (39). The i in the matrix from $\mathbb{R}_4$ and $\mathbb{L}_3$ runs from 1 to 3. Please see Appendix A.1 for the entries marked by $*$

	$\mathbb{L}_3$	$\mathbb{L}_4$
$\mathbb{R}_1$	$\begin{bmatrix} y_1^2 M_1^{-1} + y_2^2 M_{23}^{-1} & y_1^2 M_1^{-1} + y_2 y_{23} M_{23}^{-1} & y_1^2 M_1^{-1} + y_2 y_{23} M_{23}^{-1} \\ y_1^2 M_1^{-1} + y_2 y_{23} M_{23}^{-1} & y_1^2 M_1^{-1} + y_{23}^2 M_{23}^{-1} & y_1^2 M_1^{-1} + y_{23}^2 M_{23}^{-1} \\ y_1^2 M_1^{-1} + y_2 y_{23} M_{23}^{-1} & y_1^2 M_1^{-1} + y_{23}^2 M_{23}^{-1} & y_1^2 M_1^{-1} + y_{23}^2 M_{23}^{-1} \end{bmatrix}$	$\begin{bmatrix} y_1^2 M_1^{-1} & * & * \\ * & y_2^2 M_{23}^{-1} & y_2 y_3 M_{23}^{-1} \\ * & y_2 y_3 M_{23}^{-1} & y_3^2 M_{23}^{-1} \end{bmatrix}^a$
$\mathbb{R}_2$	$\begin{bmatrix} y_{12}^2 M_{12}^{-1} & y_{12}^2 M_{12}^{-1} & y_{12}^2 M_{12}^{-1} \\ y_{12}^2 M_{12}^{-1} & y_{12}^2 M_{12}^{-1} + y_3^2 M_3^{-1} & y_{12}^2 M_{12}^{-1} + y_3^2 M_3^{-1} \\ y_{12}^2 M_{12}^{-1} & y_{12}^2 M_{12}^{-1} + y_3^2 M_3^{-1} & y_{12}^2 M_{12}^{-1} + y_3^2 M_3^{-1} \end{bmatrix}^b$	$\begin{bmatrix} y_1^2 M_{12}^{-1} & y_1 y_2 M_{12}^{-1} & * \\ y_1 y_2 M_{12}^{-1} & y_2^2 M_{12}^{-1} & * \\ * & * & y_3^2 M_3^{-1} \end{bmatrix}$
$\mathbb{R}_3$	$\begin{bmatrix} y_{12}^2 M_{123}^{-1} & y_{12}^2 M_{123}^{-1} + y_{12} y_3 M_{23}^{-1} & y_{12}^2 M_{123}^{-1} + y_{12} y_3 M_{23}^{-1} \\ y_{12}^2 M_{123}^{-1} + y_{12} y_3 M_{23}^{-1} & y_{12}^2 M_1^{-1} + y_{123}^2 M_{23}^{-1} & y_{12}^2 M_1^{-1} + y_{123}^2 M_{23}^{-1} \\ y_{12}^2 M_{123}^{-1} + y_{12} y_3 M_{23}^{-1} & y_{12}^2 M_1^{-1} + y_{123}^2 M_{23}^{-1} & y_{12}^2 M_1^{-1} + y_{123}^2 M_{23}^{-1} \end{bmatrix}$	$\begin{bmatrix} y_1^2 M_{123}^{-1} & y_1 y_2 M_{123}^{-1} & y_1 y_3 M_{23}^{-1} \\ y_1 y_2 M_{123}^{-1} & y_2^2 M_{123}^{-1} & y_2 y_3 M_{23}^{-1} \\ y_1 y_3 M_{23}^{-1} & y_2 y_3 M_{23}^{-1} & y_3^2 M_{23}^{-1} \end{bmatrix}$
$\mathbb{R}_4$	$\begin{bmatrix} y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & y_1^2 M_1^{-1} + y_2^2 M_2^{-1} \\ y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & y_i^2 M_i^{-1} & y_i^2 M_i^{-1} \\ y_1^2 M_1^{-1} + y_2^2 M_2^{-1} & y_i^2 M_i^{-1} & y_i^2 M_i^{-1} \end{bmatrix}^c$	$\begin{bmatrix} y_1^2 M_1^{-1} & * & * \\ * & y_2^2 M_2^{-1} & * \\ * & * & y_3^2 M_3^{-1} \end{bmatrix}$

[a] Model E: $(\mathbb{R}_1, \mathbb{L}_4)$, $y_2 \sim y_3$
[b] Model F: $(\mathbb{R}_2, \mathbb{L}_3)$, $y_2 \ll y_3$
[c] Model G: $(\mathbb{R}_4, \mathbb{L}_3)$, $y_2 \ll y_3$

Table 4. List of phenomenologically viable models of Yukawa matrices and the light neutrino mass matrix divided by v^2 derived through the see-saw mechanism. The eigenvalues in Models A, C, and E are of $y_2 \ll y_3 \sim 1$, while those in Models B, D, F, and G are of $y_1 \ll y_2 \sim y_3 \sim 1$. The y_2 and y_3 specified in m_ν/v^2 remain for later use

Model	$\lvert\mathbb{L}\rvert$	$\mathcal{Y}_D$	$\lvert\mathbb{R}^\dagger\rvert$	$m_\nu/v^2 \sim \mathbb{L}\mathcal{Y}_D\mathbb{R}^\dagger M_\mathrm{R}^{-1}\mathbb{R}^*\mathcal{Y}_D^\mathrm{T}\mathbb{L}^\mathrm{T}$
A	$\begin{bmatrix}1&\lambda&\lambda\\\lambda&1&1\\\lambda&1&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&1&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&\rho&\rho\\\rho&1&1\\\rho&1&1\end{bmatrix}$	$\begin{bmatrix}y_1^2M_1^{-1}&\surd&\surd\\(y_1^2M_1^{-1}+y_{23}^2M_{23}^{-1})\lambda+y_1y_{23}M_{123}^{-1}\rho&y_{23}^2M_{23}^{-1}&\surd\\(y_1^2M_1^{-1}+y_{23}^2M_{23}^{-1})\lambda+y_1y_{23}M_{123}^{-1}\rho&y_{23}^2M_{23}^{-1}&y_{23}^2M_{23}^{-1}\end{bmatrix}$
B	$\begin{bmatrix}1&\lambda&\lambda\\\lambda&1&1\\\lambda&1&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&y_2&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&1&\rho\\1&1&\rho\\\rho&\rho&1\end{bmatrix}$	$\begin{bmatrix}y_1^2M_{12}^{-1}&\surd&\surd\\y_1y_2M_{12}^{-1}&y_2^2M_{12}^{-1}+y_3^2M_3^{-1}&\surd\\y_1y_2M_{12}^{-1}&y_2^2M_{12}^{-1}+y_3^2M_3^{-1}&y_2^2M_{12}^{-1}+y_3^2M_3^{-1}\end{bmatrix}$
C	$\begin{bmatrix}1&\lambda&\lambda\\\lambda&1&1\\\lambda&1&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&1&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&\rho&\rho\\\rho&1&\rho\\\rho&\rho&1\end{bmatrix}$	$\begin{bmatrix}y_1^2M_1^{-1}&\surd&\surd\\y_i^2M_i^{-1}\lambda+y_1(y_2+y_3)M_{13}^{-1}\rho&y_2^2M_2^{-1}+y_3^2M_3^{-1}&\surd\\y_i^2M_i^{-1}\lambda+y_1(y_2+y_3)M_{13}^{-1}\rho&y_2^2M_2^{-1}+y_3^2M_3^{-1}&y_2^2M_2^{-1}+y_3^2M_3^{-1}\end{bmatrix}$
D	$\begin{bmatrix}1&\lambda&\lambda\\\lambda&1&1\\\lambda&1&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&y_2&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&\rho&\rho\\\rho&1&1\\\rho&1&1\end{bmatrix}$	$\begin{bmatrix}y_1^2M_1^{-1}&\surd&\surd\\y_i^2M_i^{-1}\lambda+y_1(y_2+y_3)M_{13}^{-1}\rho&y_2^2M_2^{-1}+y_3^2M_3^{-1}&\surd\\y_i^2M_i^{-1}\lambda+y_1(y_2+y_3)M_{13}^{-1}\rho&y_2^2M_2^{-1}+y_3^2M_3^{-1}&y_2^2M_2^{-1}+y_3^2M_3^{-1}\end{bmatrix}$
E	$\begin{bmatrix}1&\lambda&\lambda\\\lambda&1&\lambda\\\lambda&\lambda&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&1&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&\rho&\rho\\\rho&1&1\\\rho&1&1\end{bmatrix}$	$\begin{bmatrix}y_1^2M_1^{-1}&\surd&\surd\\(y_1^2M_1^{-1}+y_2y_{23}M_{23}^{-1})\lambda+y_1y_2M_{123}^{-1}\rho&y_2^2M_{23}^{-1}&\surd\\(y_1^2M_1^{-1}+y_3y_{23}M_{23}^{-1})\lambda+y_1y_3M_{123}^{-1}\rho&y_2y_3M_{23}^{-1}&y_3^2M_{23}^{-1}\end{bmatrix}$
F	$\begin{bmatrix}1&1&\lambda\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&y_2&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&1&\rho\\1&1&\rho\\\rho&\rho&1\end{bmatrix}$	$\begin{bmatrix}y_{12}^2M_{12}^{-1}&\surd&\surd\\y_{12}^2M_{12}^{-1}&y_{12}^2M_{12}^{-1}+y_3^2M_3^{-1}&\surd\\y_{12}^2M_{12}^{-1}&y_{12}^2M_{12}^{-1}+y_3^2M_3^{-1}&y_{12}^2M_{12}^{-1}+y_3^2M_3^{-1}\end{bmatrix}$
G	$\begin{bmatrix}1&1&\lambda\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}y_1&&\\&y_2&\\&&1\end{bmatrix}$	$\begin{bmatrix}1&\rho&\rho\\\rho&1&\rho\\\rho&\rho&1\end{bmatrix}$	$\begin{bmatrix}y_1^2M_1^{-1}+y_2^2M_2^{-1}&\surd&\surd\\y_1^2M_1^{-1}+y_2^2M_2^{-1}&y_i^2M_i^{-1}&\surd\\y_1^2M_1^{-1}+y_2^2M_2^{-1}&y_i^2M_i^{-1}&y_i^2M_i^{-1}\end{bmatrix}$

The structures listed in Tables 2 and 3 are investigated as to whether they can predict two large mixing angles and one small mixing angle in light neutrinos. As for the large mixing angle in atmospheric neutrino oscillations, the ratio of two elements $m_{\nu 23}/m_{\nu 33}$ must not be less than of order 1. On the other hand, for the ratio $m_{\nu 12}/m_{\nu 22}$ to be not less than of order 1 is a sufficient condition for the large mixing angle in solar neutrino oscillations, because there is still room for a kind of manipulation including fine tuning so as to make the mixing angle large even without the naive ratio $m_{\nu 12}/m_{\nu 22}$. The application of the constraints from the light neutrino mass spectrum is described more fully in Appendix A.2. When $y_2 \sim y_3$, the combinations of $\mathbb{R}$ and $\mathbb{L}$ which help us predict a large mixing angle for atmospheric neutrinos is $(\mathbb{R}_1, \mathbb{L}_1)$ for Model A, $(\mathbb{R}_4, \mathbb{L}_1)$ for Model C, or $(\mathbb{R}_1, \mathbb{L}_4)$ for Model E, where $\mathbb{R}_i$ and $\mathbb{L}_i$ with $i = 1$–4 are defined in (39). The solutions $\mathbb{R}_2$ or $\mathbb{R}_3$ are not included since the out-of-equilibrium condition with $y_2 \sim y_3$ ruled out a large angle at R_{12}, as shown in Table 1. When $y_2 \ll y_3$, $(\mathbb{R}_2, \mathbb{L}_1)$ for Model B, $(\mathbb{R}_4, \mathbb{L}_1)$ for Model D, $(\mathbb{R}_2, \mathbb{L}_3)$ for Model F, $(\mathbb{R}_4, \mathbb{L}_3)$ for Model G can predict a large mixing angle for atmospheric neutrinos and the CHOOZ bound of reactor neutrinos. See the appendix.

Thus, the structures of Yukawa matrices that satisfy three constraints:

(i) the out-of-equilibrium condition of lepton number-violating processes,

(ii) the upper bound of the branching ratio of the rare decay $\mu \to e\gamma$, and

(iii) a large mixing angle in atmospheric neutrino oscillations, must be one of the seven combinations, Models A to G in Tables 2 and 3, which establish the relative dominance of y_2 and y_3, and the approximate magnitudes of the transformation matrices, $\lvert\mathbb{R}\rvert$ and $\lvert\mathbb{L}\rvert$. In Table 4 is the list of models of Yukawa matrices and the light neutrino mass matrix divided by v^2. The models are specified in terms of two unitary transformations and the relative

Table 5. The specification of constraints which in fact rule out certain types of Yukawa matrices represented by a $\mathbb{R}$, a $\mathbb{L}$, and the comparison of y_2 and y_3. The C1, C2, C3, and C4 are the indices of the constraints explained in Sect. A.2. For example, the Yukawa matrix characterized by $\mathbb{R}_1, \mathbb{L}_1$ and $y_2 \ll y_3$ is forbidden due to the constraint C4

	$\mathbb{L}_1$	$\mathbb{L}_2$	$\mathbb{L}_3$	$\mathbb{L}_4$
$\mathbb{R}_1$				
$y_2 \ll y_3$	C4	C3	C4	C3
$y_2 \sim y_3$	Model A	C2 C3	C2 C3	Model E
$\mathbb{R}_2$				
$y_2 \ll y_3$	Model B	C3	Model F	C3
$y_2 \sim y_3$	C1	C1 C2 C3	C1 C2	C1 C3
$\mathbb{R}_3$				
$y_2 \ll y_3$	C4	C3	C4	C3
$y_2 \sim y_3$	C1	C1 C2 C3	C1 C2	C1
$\mathbb{R}_4$				
$y_2 \ll y_3$	Model C	C3	Model G	C3
$y_2 \sim y_3$	Model D	C2 C3	C2 C3	C3

dominance in the eigenvalues as summarized from Tables 2 and 3. The notations are introduced in (39), (41), (42), and (A.1)–(A.3). An entry marked with $\surd$ represents the corresponding symmetric elements. There are two types of spectrum of eigenvalues, $y_1 \ll y_2$, $y_1 \sim y_2 \ll y_3 \sim 1$ and $y_1 \ll y_2 \sim y_3 \sim 1$, which correspond to Case (a), (c) and Case (b) as introduced for Table 1, respectively.

In the appendix, Table 5 describes in detail which constraints keep certain entries in Tables 2 and 3 from being eligible models. Table 5 in Appendix A.2 shows obviously that the cases prohibited by the constraint (ii) are already prohibited by the constraint (iii). The constraint (ii) turns out to be a redundant one in checking the eligibility of the Yukawa matrix to known leptonic phenomenology. In other words, the structure of the Yukawa matrix which can derive the light neutrino mass matrix through the see-saw mechanism always appear to give rise to a $\mathrm{Br}(\mu \to e\gamma)$ below the present experimental limit in SUSY theories with large $\tan\beta$ and universal slepton mass at GUT scale. It is possible to pull out a constraint to narrow down the estimation of heavy neutrino masses from LFV through the see-saw mechanism.

5 Models and a link between leptogenesis and low-energy neutrinos

5.1 Application of bi-unitary parametrization

The light neutrino mass matrix defined through the see-saw mechanism in (37) is diagonalized by

$$\mathbb{V} m_{\mathrm{L}} \mathbb{V}^{\mathrm{T}} = -v_2^2 \mathbb{L} \mathcal{Y}_D \mathbb{R}^{\dagger} M_{\mathrm{R}}^{-1} \mathbb{R}^{*} \mathcal{Y}_D^{\mathrm{T}} \mathbb{L}^{\mathrm{T}}, \quad (43)$$

where $\mathbb{V}$ is the MNS light neutrino transformation matrix and m_{L} is a diagonal matrix with mass eigenvalues m_i. The equation of 3×3 symmetric complex matrices in (43) consists of 12 equations of parameters, while its right-hand side consists of 18 unknowns: three angles and three phases in $\mathbb{L}$, three angles and three phases in $\mathbb{R}$, three eigenvalues in $\mathcal{Y}_D$, and three heavy neutrino masses in M_{R}. Once one makes a choice out of the seven models model A to model G specified in Tables 2 and 3, one can draw the boundary of eligible parameter space starting with the six angles, large or small. Even though it is supposed that the number of the free parameters reduces to 12 for 12 equations, considering six angles in $|\mathbb{R}|$ and $|\mathbb{L}|$ fixed, solving (43) yet embeds a great deal of ambiguity, since the CP phases in the MNS matrix and individual mass eigenvalues on the left-hand side can barely be said to be fixed.

There are a few approaches to make improvements in solving the see-saw mechanism depending on the models. First, one can rely on theories with additional symmetries to reduce the number of parameters. For example, in the case of Model G, a flavor symmetry like $U(1)$ can be utilized to simplify the mechanism in (43), because the charges of the flavor symmetry assigned to $\nu_{\mathrm{R}i}$ (ℓ_i) can result in a common constraint on the side of right (left) mixing in the Yukawa matrix and the structure of the right (left) neutrino mass matrix. The structure of the Yukawa matrix in Model E can fit in a theory of grand-unified-type symmetry between leptons and quarks, if $\mathbb{L}_1$ is close to the CKM matrix. Among the eigenvalues of the Yukawa matrix and three heavy masses, there are a few parameters which can be fixed by hand, motivated from a higher-rank symmetry. They can be y_3 of order 1 and/or M_3 of the order of the GUT scale. The $y_3 \sim 1$ has been used in a number of times after (27), as well as $y_2 \sim y_3$ in particular models.

Another possible approach can be collecting more phenomenological constraints involved in neutrino Yukawa couplings to diminish the number of the eligible structures of the Yukawa matrix, aiming at the survival of only one natural structure. The classification derived based on bi-unitary transformations is useful in both approaches. That is, the models parameterized using a bi-unitary transformation of the Yukawa matrix can easily be tested for the compatibility to additional symmetries as proposed in the previous paragraph. The bi-unitary parametrization is useful also to examine the eligibility to additional phenomenology. In describing a process, the loop contributions of the Yukawa couplings consist of a hermitian operator $\mathcal{Y}^{\dagger}\mathcal{Y}$ or $\mathcal{Y}\mathcal{Y}^{\dagger}$. It is always possible to rephrase the operator in terms of the minimal number of parameters, the pair of $(\mathcal{Y}_D, \mathbb{R})$ or the pair of $(\mathcal{Y}_D, \mathbb{L})$ without any mixing of the two transformations. Thus, application of an additional constraint will simply be the re-examination of the seven models characterized by eigenvalues of the Yukawa matrix, and left and right transformations.

5.2 A model of neutrino masses

The description of leptogenesis requires precise understanding of the CP phases in the Yukawa matrix as well as the CP phases in the light neutrino mass matrix in order to secure a sufficient amount of CP asymmetry from the decays of heavy Majorana neutrinos. In the lepton sector, however, it is not practical to regard the CP phases at low energy as known parameters. Rather, in analogy to the quark sector, the size of the real mixing angle can be realized as the source of a certain amount of CP violation before the derivation of detailed imaginary parts. Many models proposed so far took simply an order 1 contribution from the CP phase.

As far as the detailed discussion of CP phases is set aside, Models A, D, and E with a large angle at R_{23} can provide an optimal scenario for a sufficient amount of CP asymmetry, and accordingly lepton asymmetry, (5). There are a number of cases listed in Table 1 which can enhance the CP asymmetry ϵ_1, if $|R_{23}|$ becomes close to 1, so as to reach its maximum value:

$$\epsilon_1 \sim 10^{-1} y_3^2 |R_{23}|^2 \frac{M_1}{M_2} \sim 10^{-1} \frac{M_1}{M_2}. \tag{44}$$

It is worthy of attention that large mixing angles in low energy originate entirely from the large mixings in heavy neutrinos in Model E.

The Yukawa matrix in Model E is

$$|\mathcal{Y}_N| \equiv \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & 1 \\ \rho & 1 & 1 \end{pmatrix}, \tag{45}$$

where $y_2 \sim y_3 \sim 1, y_1 \ll y_2$. It is possible that Model E reduces to the model with two-generations of heavy neutrinos as the large mixing angles become maximal and the degeneracy in the eigenvalues becomes exact. In such a limit, the leading contribution to ϵ_1 no longer comes from R_{23}. The matrix in (45) was introduced in the basis in which the mass matrix of right-handed neutrinos is diagonal. Now, a new basis can be chosen in such a way that the Yukawa matrix is symmetric. Let the Yukawa matrix in Model E in the new basis be

$$|\mathcal{Y}'_N| \equiv \begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho' & \rho' \\ \rho' & 1 & \rho' \\ \rho' & \rho' & 1 \end{pmatrix}. \tag{46}$$

Then, the right-handed neutrino mass matrix in (43) will have a structure with a large mixing angle in the new basis,

$$M'_{\mathrm{R}} \sim \mathbb{R}' M_{\mathrm{R}} \mathbb{R}'^{\mathrm{T}}, \qquad \mathbb{R}' \equiv \mathbb{R}^*_{\mathrm{new}} \mathbb{R}^{\mathrm{T}}_{\mathrm{old}}, \tag{47}$$

where $\mathbb{R}_{\mathrm{old}}$ is the right transformation to diagonalize the Yukawa matrix from the old basis in (45), while $\mathbb{R}_{\mathrm{new}}$ is the one for the diagonalization from the new basis in (46). Model E can represent a model in left–right symmetry; in other words, the model shows a well-balanced correspondence between low energy and high energy over the see-saw mechanism, since the Yukawa matrix is symmetric and two Majorana neutrino mass matrices are very much in analogy.

The difference of the lepton sector from the quark sector was recognized from the large mixing angles. In this scenario, the dissimilarity in the two sectors can be understood as originating from the existence of massive singlet neutrinos, whereas there is nothing like that in the quark sector. There might have been dynamics of the heavy neutrinos in the early universe to generate the large mixing angles which later become the source of the large mixing angles of the light neutrinos. It is shown in (44) that the leptogenesis is optimized. Equations (46) and (47) help the see-saw mechanism look more meaningful with this model in a theory with left–right symmetry.

5.3 Leptogenesis and neutrino large mixing angles

Before concluding this section, we give some thought to the role of the see-saw mechanism, expressing CP asymmetry in terms of light neutrino parameters in a model-independent way. Using solutions to (43) for a chosen model, one can express the CP asymmetry in a proper way to replace the heavy neutrino masses in terms of low-energy neutrino observables and parameters in the Yukawa matrix. The diagonal neutrino mass matrix in (43) can be rephrased as

$$m_{\mathrm{L}} = -v_2^2 \mathbb{V}^{\mathrm{T}} \tilde{\mathcal{Y}}_N M_{\mathrm{R}}^{-1} \mathbb{P}^2 \tilde{\mathcal{Y}}_N^{\mathrm{T}} \mathbb{V}, \tag{48}$$

where $\tilde{\mathcal{Y}}_N \equiv \mathcal{Y}_N \mathbb{P}^\dagger$ is defined in such a way that the phases in $\mathbb{P}$ do not appear explicitly in $\tilde{\mathcal{Y}}_N$ as hidden in (20). Alternatively,

$$M_k \mathbb{P}_k^{-2} = -v_2^2 \sum_{k'} \frac{1}{m_{k'}} \left(\mathbb{V}^{\mathrm{T}} \tilde{\mathcal{Y}}_N\right)_{k'k}^2, \tag{49}$$

and the imaginary parts of the ones inverse to those are obtained as

$$\frac{\sin 2\phi_1}{M_2} = \frac{1}{v_2^2} \left(\sum_k \frac{1}{m_k} \mathrm{Im} \left[\mathbb{V}^{\mathrm{T}} \tilde{\mathcal{Y}}_N\right]_{k2}^2 \right)^{-1}, \tag{50}$$

$$\frac{\sin 2\phi_2}{M_3} = \frac{1}{v_2^2} \left(\sum_k \frac{1}{m_k} \mathrm{Im} \left[\mathbb{V}^{\mathrm{T}} \tilde{\mathcal{Y}}_N\right]_{k3}^2 \right)^{-1}, \tag{51}$$

while

$$M_1 = \frac{1}{v_2^2} \sum_k \frac{1}{m_k} \left[\mathbb{V}^{\mathrm{T}} \tilde{\mathcal{Y}}_N\right]_{k1}^2. \tag{52}$$

When the value of the rectangular brackets of the above equation is of order 1, the term with m_1^{-1} is guaranteed to be a significantly leading contribution compared to the other terms with m_2^{-1} or m_3^{-1} inside the summation, resulting in the enhancement in the scale of M_1. In [17], it was examined that large mixing angles in the MNS matrix can predict the small dilution mass preferred by the out-of-equilibrium condition in a model. On the right-hand side of (21), $\sin 2\phi_1\,(M_1/M_2)$ and $\sin 2\phi_2\,(M_1/M_3)$ can be re-parameterized only in terms of the Yukawa matrix and light neutrino masses and mixings using (51) and (52). The enhancement of M_1 contributed from large mixing angles in (52) results in the enhancement in the amount of CP violation in (21).

6 Summary

The CP asymmetry and the out-of-equilibrium condition of a lepton flavor violating process are analyzed in terms of eigenvalues and right mixing angles. In the out-of-equilibrium condition, there are significant implications. The mixing element $|\mathbb{R}_{13}|$ should be small with an upper bound, (28), and the two eigenvalues y_1 and y_3 cannot be of similar size, (29). Depending on which term in the expression of the condition actually is leading in the condition or depending on whether eigenvalues are in hierarchy or of similar size, the magnitude of the right transformation is characterized differently, as shown in Table 1. Examining all the possible models, we found the model independent maximal CP asymmetry to be $10^{-1}M_1/M_2$; see (32). Besides the thermal out-of-equilibrium condition, we examined also the upper bound of a LFV decay and mixing angles in neutrino oscillations to constrain all kinds of parameters, i.e., left and right mixing elements and eigenvalues. We found that only the seven models can characterize phenomenologically eligible Yukawa matrices.

The constraint from the LFV did not exclude any choice of the parameter region in addition to the category excluded by the two other constraints. In other words, the survival of only seven models is the outcome of the application of two constraints: the out-of-equilibrium condition and the data of the neutrino oscillations. In Table 5, the determination of eligible models from the application of certain constraints is specified. The bi-unitary parametrization is useful for a model test of the compatibility to additional symmetries and the eligibility to additional phenomenology. Model E is appealing in a sense that the size of the CP asymmetry is maximized and the number of free parameters can be reduced based on left–right symmetry.

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Appendix

A Remarks on Tables 2 and 3

A.1 Recovery of the small-angle contribution in transformations

In Tables 2 and 3, the elements of the matrices determined from small angles in the transformations are not specified. (They are indicated by $*$.) Models A, C, and E include elements, the leading order of which is first order in λ or ρ, while the other models have all zeroth leading orders in λ and ρ. The small values are not all of the same order of magnitude in a transformation matrix. However, different small angles are not distinguished because a comparison of their sizes is not necessary to examine the eligibility of a model to the constraints given in this paper. With the ρ and λ in the $\mathbb{R}$ and $\mathbb{L}$, respectively, the possible symmetric structures of the matrix m_ν/v_2^2 of Models A, C, and E will be as follows.

Model A:

$$\begin{bmatrix} y_1^2M_1^{-1} & \surd & \surd \\ \begin{Bmatrix} \left(y_1^2M_1^{-1}+y_{23}^2M_{23}^{-1}\right)\lambda \\ +y_1y_{23}M_{123}^{-1}\rho \end{Bmatrix} & y_{23}^2M_{23}^{-1} & \surd \\ \begin{Bmatrix} \left(y_1^2M_1^{-1}+y_{23}^2M_{23}^{-1}\right)\lambda \\ +y_1y_{23}M_{123}^{-1}\rho \end{Bmatrix} & y_{23}^2M_{23}^{-1} & y_{23}^2M_{23}^{-1} \end{bmatrix}. \quad (A.1)$$

Model C:

$$\begin{bmatrix} y_1^2M_1^{-1} & \surd & \surd \\ \begin{Bmatrix} y_i^2M_i^{-1}\lambda+ \\ y_1\left(y_2+y_3\right)M_{13}^{-1}\rho \end{Bmatrix} & \begin{Bmatrix} y_2^2M_2^{-1}+ \\ y_3^2M_3^{-1} \end{Bmatrix} & \surd \\ \begin{Bmatrix} y_i^2M_i^{-1}\lambda+ \\ y_1\left(y_2+y_3\right)M_{13}^{-1}\rho \end{Bmatrix} & \begin{Bmatrix} y_2^2M_2^{-1}+ \\ y_3^2M_3^{-1} \end{Bmatrix} & \begin{Bmatrix} y_2^2M_2^{-1}+ \\ y_3^2M_3^{-1} \end{Bmatrix} \end{bmatrix}. \quad (A.2)$$

Model E:

$$\begin{bmatrix} y_1^2M_1^{-1} & \surd & \surd \\ \begin{Bmatrix} \left(y_1^2M_1^{-1}+y_2y_{23}M_{23}^{-1}\right)\lambda \\ +y_1y_2M_{123}^{-1}\rho \end{Bmatrix} & y_2^2M_{23}^{-1} & \surd \\ \begin{Bmatrix} \left(y_1^2M_1^{-1}+y_3y_{23}M_{23}^{-1}\right)\lambda \\ +y_1y_3M_{123}^{-1}\rho \end{Bmatrix} & y_2y_3M_{23}^{-1} & y_3^2M_{23}^{-1} \end{bmatrix}, \quad (A.3)$$

where $i = 1$–3 and the entry marked by $\surd$ represents the corresponding symmetric elements.

The 1-2 sector in the light neutrino mass needs to be examined to see whether a model is consistent with the large mixing angle of solar neutrino oscillations. In order for a mass matrix to have the large mixing angle between the first and the second generations, the 1-2 element of the matrix $m_{\nu 12}$ should not be smaller than the 1-1 element $m_{\nu 11}$. If one takes a hierarchical neutrino mass spectrum: $m_2 \sim \sqrt{\Delta m_\odot^2}$ and $m_3 \sim \sqrt{\Delta m_{\rm atm}^2}$, then the structure of neutrino mass matrix should require

$$\frac{m_{\nu 12}}{m_{\nu 22}} \sim \sqrt{\frac{\Delta m_\odot^2}{\Delta m_{\rm atm}^2}}, \quad (A.4)$$

i.e., the ratio between the two elements should be suppressed by one order of magnitude [33]. The m_ν/v_2^2 matrices for the three models in (A.1)–(A.3) show that there can be a parameter region which allows a large mixing angle solution to be obtained for solar neutrino oscillations as well as the large mixing angles of atmospheric neutrinos satisfying the CHOOZ bound of the reactor neutrinos.

A.2 Application of constraints

The exclusion of certain types of Yukawa structures is explained in Table 5 by specifying the constraint violated by each particular structure. The constraint C1 is equivalent to the constraint (i) listed in Sect. 4. The out-of-equilibrium condition of the lepton number-violating process, (24), does not allow a structure of the Yukawa matrix to occur with both $y_2 \sim y_3$ and $R_{12} \sim 1$.

The constraint C2 is equivalent to the constraint (ii) in Sect. 4. The suppression in $\left(\mathcal{Y}_N\mathcal{Y}_N^\dagger\right)_{21}$ for LFV in (33) does not allow for a structure of the Yukawa matrix with both $y_2 \sim y_3$ and $L_{12} \sim 1$.

The constraint C3 is equivalent to the constraint (iii) in Sect. 4. The phenomenological constraints for the light neutrino experiments require the structure of the mass matrix to have $m_{\nu 13} \ll m_{\nu 23} \sim m_{\nu 33}$.

The last constraint C4 is the requirement for the Yukawa matrix not to involve fine tuning. It is not appropriate to assume any possible fine tuning in the Yukawa structure since the comparison of parameters was done whether it is hierarchical or nearly degenerate, whereas it is unavoidable to assume some possible fine tuning in the consideration with light neutrino masses due to the coexistence of large mixing angles and the hierarchical mass spectrum in its symmetric structure. In the Yukawa matrices, if $y_2 \ll y_3$ when $L_{23} \sim 1$ and $R_{23} \sim 1$, such a small eigenvalue y_2 can be obtained only by fine tuning. For that reason, the cases with $L_{23} \sim R_{23} \sim 1, y_2 \ll y_3$ are ruled out. In actual model construction with respect to the data, however, the structures restricted by C4 can be regarded rather positively, if the fine tuning is the only way to accommodate the data. If one allows tuning to be done with $L_{23} \sim R_{23} \sim 1, y_2 \ll y_3$, there can be four more eligible models, which are marked by C4 in Table 5.

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